

Mathematics

I would characterize having a good mathematical background as the ability to extract the problem from a context, use mathematics to solve the problem and then interpret the solution back into the context. This is an important skill.

Faculty Viewpoint

Knowledge & Skills Foundations

As in many other disciplines, incoming students in mathematics are expected to bring a combination of hands-on skills and conceptual understanding. Entering students need to know basic mathematical concepts—computation, algebra, trigonometry, geometry—so that they have the tools to work with increasingly complex conceptual mathematical and quantitative procedures and analyses in their college courses.

Understanding Mathematics

Successful students approach mathematical problems as they would an investigation. They ask questions, reflect and revisit their solutions with this idea in mind: It is important how one reaches a solution and why a solution works. Problem solving involves analytical processes and sets of skills. These skills include, among others, the following:

1. Thinking conceptually, not just procedurally, about mathematics. Successful students understand the relationships that exist between mathematical concepts and that formulas do not function in a vacuum. They perceive mathematics as a way of understanding, a thinking process and not a collection of detached procedures to be learned and applied separately.
2. Using logical reasoning and common sense to work on and find mathematical solutions. Successful students are able to provide supporting evidence to construct compelling arguments to explain processes and solutions. They check their solutions through visualization, so that they can see whether their findings make sense or not.
3. Using experimental thinking, inquisitiveness and a willingness to investigate the steps used to reach a solution. Successful students understand there can be multiple approaches to solving a problem.
4. Taking risks and accepting failure as part of the learning process. When students do not find the correct answer to a problem, it is an opportunity to revisit the procedures they used, try new ones and ask further questions. Finding a solution may be only vaguely logical. Verifying a solution should be rigorously logical.
5. The ability to use formulas and algorithms of computation. A lack of facility with computation and formulas encumbers the analytical process.

Problem Solving, Technology and Communication

Problem solving is central to the teaching and learning of mathematics. The step-by-step approach is the best way to solve math

30 Mathematics

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problems and draw parallels and connections from various problems. Mathematical problem solving involves logical reasoning; it is important to explore the reasons why step two follows step one. Successful students understand the process of modifying, adapting and combining mathematical tools to find new ways to reach a solution. They also need to question results until they can explain their answers and defend them. Technology is important and relevant to the understanding of mathematics. However, students need to be aware of its limitations and recognize that calculators are tools that assist but do not replace the thinking process. A graphing calculator can be a tool to deepen understanding of functions and as a way to represent them visually. At the same time, successful students can identify whether the calculator's answers are reasonable in light of their own calculations.

Mathematics is the language of the sciences, and thus fluency in this language is a basic skill. Students prepared for college-level study are comfortable with mathematic terminology and use it appropriately. It is crucial to understand that formulas and symbols provide precise statements of often vaguely posed problems. Different interpretations of a problem may lead to different mathematical models and analyses. Students must pay attention to the wording of problems and move with ease between the symbolic representation of a problem and its verbal presentation.

Students need to understand that process is also important in mathematics. They tend to put too much emphasis on the answer.

Faculty Viewpoint

While mathematics is a type of language, the study of math also requires solid verbal skills. In mathematics, students are expected to write with clarity and cohesiveness. A poorly written solution is often an indication of confused thinking. While clarity in writing is the best way to convey information to others, it is also an important indication that one understands the problem.

Orientation Towards Learning

Relating mathematical abstractions to life outside of mathematics courses is a highly useful skill. Students who do well in mathematics classes are prepared to translate real situations into mathematical representation and, conversely, extract meaning from mathematical expression. They understand when mathematics generalizes and when it is specific, and recognize the importance of abstraction and generalization as they learn and do mathematics.

Often, college-level mathematics courses require that students work in groups. While it is important to be able to work effectively with peers, students must also develop the skills necessary to approach mathematical problems on their own; independent of classes, group projects and work environments. Doing so will help students get the most from group activities. Both situations are valuable—as are the skills to work within them.

Students often experience anxiety when confronted with a mathematical problem, even when encountering mathematical terminology. Persistence is invaluable in the quest for correct answers to a problem, and it is vital to tolerate ambiguity on the road to solution. Interestingly enough, some faculty expressed a concern about students being too confident in their perceived knowledge and skills. Students are sometimes naively confident, preventing themselves from engaging in the mathematical process, finding other solutions and estimating or questioning the viability of their results.

Mathematical problems rarely have instant or quick solutions and often require long periods of time before a solution can be found. Sustained inquiry—engaging in the process for more than a short time—is an important part of the process when solving a problem or writing an exam. Successful students understand that math is an academic activity that requires time, sustained engagement, patience and persistence.

When students practice multiplying and adding by hand for a long time, they get a feel for what numbers are. Now, by overusing the calculator, simple operations are gone. Mental calculations help develop a feel for math. Going to the calculator too soon is a problem.

Faculty Viewpoint

Mathematics Standards

*= Items with an asterisk are those expected of students who plan to major in these fields of study (mathematics, computer science, statistics).

I. Computation

A. Successful students know basic mathematical operations. They:

- A.1. apply arithmetic operations with fractions and integers (e.g., add and subtract by finding a common denominator, multiply and divide, reduce and perform long division without a calculator).
- A.2. use exponents and scientific notation.
- A.3. use radicals correctly.
- A.4. understand relative magnitude.
- A.5. calculate using absolute value.
- A.6. use the correct order of arithmetic operations, particularly demonstrating facility with the Distributive Law.





A.7.* know terminology for complex numbers, integers, rational numbers, irrational numbers and complex numbers.

B. Successful students know and carefully record symbolic manipulations. They:

B.1. understand the uses of mathematical symbols as well as the limitations on their appropriate uses (e.g., equal signs, parentheses, superscripts and subscripts).

C. Successful students know and demonstrate fluency with mathematical notation and computation. They:

C.1. correctly perform addition, subtraction, multiplication and division that includes variables.

C.2. perform appropriate basic operations on sets (e.g., union, intersection, elements of, subsets and complement).

C.3. use alternative symbolic expressions, particularly alternatives to x (e.g., letters of the Greek alphabet that do not already have specific scientific or mathematical meanings).

II. Algebra

A. Successful students know and apply basic algebraic concepts. They:

A.1. use the distributive property to multiply polynomials.

A.2. know how to compose and decompose functions and how to find inverses of basic functions.

A.3. simplify and perform basic operations on rational expressions, including finding common denominators (e.g., add, subtract, multiply and divide).

A.4. understand exponents, roots and their properties [e.g., $(x^2)(x^3)=x^5$ and $(\sqrt{x})^3 = x^{3/2}$].

- A.5. know basic theorems of exponents and roots.
- A.6.* understand logarithms (to bases 2, 10 and e) and their properties.
- A.7.* divide low degree polynomials (e.g., long division).
- A.8.* know basic theorems of logarithms.
- A.9.* factor polynomials (e.g., difference of squares, perfect square trinomials, difference of two cubes and trinomials such as $x^2 + 3x + 2$).

B. Successful students use various appropriate techniques to solve basic equations and inequalities. They:

- B.1. solve linear equations and absolute value equations.
- B.2. solve linear inequalities and absolute value inequalities.
- B.3. solve systems of linear equations and inequalities using algebraic and graphical methods (e.g., substitution, elimination, addition and graphing).
- B.4. solve quadratic equations using various appropriate methods while recognizing real solutions. This includes:
 - B.4a. factoring.
 - B.4b. completing the square.
 - B.4c. the quadratic formula.

C. Successful students distinguish between and among expressions, formulas, equations and functions. They:

- C.1. know when it is possible to simplify, solve, substitute or evaluate equations and expressions and when it is not possible. For example, expand, but do not solve, the expression $(x+3)(x+1)$; substitute $a = 3$, $b = 4$ into the formula $a^2 + b^2 = c^2$;

solve the equation $0 = (x+3)(x+1)$; or evaluate the function $f(x) = (x+3)(x+1)$ at $x = -1$.

- C.2. understand that the concept of a function has a specific definition beyond being a type of algebraic expression.
- C.3. represent functions, patterns and relationships in different ways (e.g., statements, formulas and graphs).
- C.4. understand the algebraic language and notation for functions (e.g., domain and range).
- C.5. understand a variety of functions (e.g., polynomial, rational, exponential, logarithmic and trigonometric) and properties of each.

D. Successful students understand the relationship between equations and graphs. They:

- D.1. understand basic forms of the equation of a straight line and how to graph the line without the aid of a calculator.
- D.2. understand the basic shape of a quadratic function and the relationships between the roots of the quadratic and zeroes of the function.
- D.3. know the basic shape of the graph of exponential and log functions, including exponential decay.

E. Successful students understand algebra well enough to apply it procedurally and conceptually to a range of common problems. They:

- E.1. recognize which type of expression best fits the context of a basic application (e.g., linear equation to solve distance/time problems; quadratic equation to explain the motion of a falling object; or compound interest as an exponential function).

F. Successful students demonstrate the ability to work with formulas and symbols algebraically. They:

- F.1.* know formal notation (e.g., sigma notation and factorial notation).
- F.2.* know arithmetic and geometric progressions and series.

III. Trigonometry

A. Successful students know and understand basic trigonometric principles. They:

- A.1. know the definitions of sine, cosine and tangent using right triangle geometry and similarity relations.
- A.2. understand the relationship between a trigonometric function in standard form and its corresponding graph (e.g., domain, range, amplitude, period, phase shift and vertical shift).
- A.3. understand periodicity and recognize graphs of periodic functions, especially the trigonometric functions.

- A.4.* know and use identities for sum and difference of angles [e.g., $\sin(x \pm y)$, $\cos(x \pm y)$] and use double and half angle formulas.

IV. Geometry

A. Successful students understand and use both basic plane and solid geometry. They:

- A.1. know properties of similarity, congruence and parallel lines cut by a transversal.
- A.2. know how to figure area and perimeter of basic figures.
- A.3. understand the ideas behind simple geometric proofs and are able to develop and write simple geometric proofs (e.g., the Pythagorean theorem; that there are 180 degrees in a triangle; and that the area of a triangle is half the base times the height).
- A.4. solve problems involving proofs through the use of geometric constructions.
- A.5. use similar triangles to find unknown angle measurements and lengths of sides.



- A.6. visualize solids and surfaces in three-dimensional space (e.g., recognize the shape of a box based on a two-dimensional representation of its surfaces; and recognize the shape of a cone based on a two-dimensional representation of its surface).
- A.7. know basic formulas for volume and surface area for three-dimensional objects.

B. Successful students know analytic (i.e., coordinate) geometry. They:

- B.1. know geometric properties of lines (e.g., slope and midpoint of a line segment).
- B.2. know the formula for the distance between two points.
- B.3. solve mathematical and real-world problems (e.g., ladders, shadows and poles) that involve the properties of special right triangles with the Pythagorean theorem and its converse.
- B.4.* recognize geometric translations and transformations algebraically.

C. Successful students understand basic relationships between geometry and algebra. They:

- C.1. know that geometric objects and figures can also be described algebraically (e.g., $ax + by = c$ is the standard form of a line).
- C.2. know the algebra and geometry of circles.
- C.3.* know the algebra and geometry of parabolas and ellipses as a prerequisite to the study of calculus.
- C.4.* use trigonometry for examples of the algebraic/geometric relationship, including Law of Sines/Cosines.



V. Mathematical Reasoning

A. Successful students know important definitions, why definitions are necessary and are able to use mathematical reasoning to solve problems. They:

- A.1. use inductive reasoning in basic arguments.
- A.2. use deductive reasoning in basic arguments.
- A.3. use geometric and visual reasoning.
- A.4. use multiple representations (e.g., analytic, numerical and geometric) to solve problems.
- A.5. learn to solve multi-step problems.
- A.6. use a variety of strategies to revise solution processes.
- A.7. understand the uses of both proof and counterexample in problem solutions and are able to conduct simple proofs.

- A.8. are familiar with the process of abstracting mathematical models from word problems, geometric problems and applications and are able to interpret solutions in the context of these source problems.

B. Successful students are able to work with mathematical notation to solve problems and to communicate solutions. They:

- B.1. translate simple statements into equations (e.g., "Bill is twice as old as John" is expressed by the equation $b=2j$).
- B.2. understand the role of written symbols in representing mathematical ideas and the precise use of special symbols of mathematics.

C. Successful students know a select list of mathematical facts and know how to build upon those facts (e.g., Pythagorean theorem; formulas for perimeter, area, volume; and quadratic formula).

D. Successful students know how to estimate. They:

- D.1. are able to convert between decimal approximations and fractions.
- D.2. know when to use an estimation or approximation in place of an exact answer.
- D.3. recognize the accuracy of an estimation.
- D.4. know how to make and use estimations.

E. Successful students understand the appropriate use as well as the limitation of calculators. They:

- E.1. recognize when the results produced are unreasonable or represent misinformation.

- E.2.* use calculators for systematic trial-and-error problem solving.
- E.3.* plot useful graphs.

F. Successful students are able to generalize and to go from specific to abstract and back again. They:

- F.1. determine the mathematical concept from the context of an external problem, solve the problem and interpret the mathematical solution in the context of the problem.
- F.2. know how to use specific instances of general facts, as well as how to look for general results that extend particular results.

G. Successful students demonstrate active participation in the process of learning mathematics. They:

- G.1. are willing to experiment with problems that have multiple solution methods.
- G.2. demonstrate an understanding of the mathematical ideas behind the steps of a solution, as well as the solution.
- G.3. show an understanding of how to modify patterns to obtain different results.
- G.4. show an understanding of how to modify solution strategies to obtain different results.
- G.5. recognize when a proposed solution does not work, analyze why and use the analysis to seek a valid solution.

H. Successful students recognize the broad range of applications of mathematical reasoning. They:

- H.1. know that mathematical applications are used in other fields (e.g. carbon dating, exponential growth, amortization tables, predator/prey models, periodic motion and the interactions of waves).



H.2. know that mathematics has played (and continues to play) an important role in the evolution of disciplines as diverse as science, engineering, music and philosophy.

A.3.* understand curve-fitting techniques (e.g., median-fit line and regression line) for various applications (e.g., making predictions).

VI. Statistics**

A. Successful students apply concepts of statistics and data analysis in the social sciences and natural sciences. They:

- A.1. represent data in a variety of ways (e.g., scatter plot, line graph and two-way table) and select the most appropriate.
- A.2. understand and use statistical summaries data (e.g., standard deviation, range and mode).

** The majority of math participants indicated that knowledge of statistics is not necessarily a prerequisite for success in most entry-level university mathematics courses. However, participants in other disciplines identified knowledge of statistics as important to success in some entry-level courses in the social sciences (e.g., economics) and sciences (e.g., biology and ecology). Statistics is being included within mathematics for organizational convenience, but should not be interpreted as equivalent to the other five areas of mathematical knowledge and skill for university success in terms of its importance in entry-level college mathematics courses. Statistics standards also appear in the natural sciences and social sciences.